Second quantized fermions with half integer spins

Understanding Nature with the Spin-Charge-Family theory

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Some publications:

- *Phys. Lett.* **B 292**, 25-29 (**1992**), *J. Math. Phys.* **34**, 3731-3745 (**1993**), *Mod. Phys. Lett.* **A 10**, 587-595 (**1995**), *Int. J. Theor. Phys.* **40**, 315-337 (**2001**),
- *Phys. Rev.* **D 62** (04010-14) (**2000**), *Phys. Lett.* **B 633** (**2006**) 771-775, **B 644** (**2007**) 198-202, **B** (**2008**) 110.1016, *JHEP* **04** (**2014**) 165, *Fortschritte Der Physik-Progress in Physics*, **(2017) with** H.B.Nielsen,
- *Phys. Rev.* **D 74** 073013-16 (**2006**), **with** A. Borštnik Bračič,
- *New J. of Phys.* **10** (**2008**) 093002, arxiv:1412.5866, **with** G.Bregar, M.Breskvar, D.Lukman,
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- *New J. of Phys.* (**2011**) 103027, *J. Phys. A: Math. Theor.* **45** (**2012**) 465401, *J. Phys. A: Math. Theor.* **45** (**2012**) 465401, *J. of Mod. Phys.* **4** (**2013**) 823-847, arxiv:1409.4981, **6** (**2015**) 2244-2247, *Phys. Rev.* **D 91** (**2015**) 6, 065004, . *J. Phys.: Conf. Ser. 845 01* **IARD 2017**, *Eur. Phys. J.C.* **77** (2017) 231, **[arXiv:1082.05554v4]**

Second quantized fermions with half integer spins

- **Algebras in Clifford space** in $d \geq (13+1)$, if used to describe the internal degrees of freedom of **fermions** — spins and charges of quarks and leptons and antiquarks and antileptons – offer the **anticommutation relations** among creation and annihilations operators for fermions **without postulating the anticommutation relations**. Correspondingly these algebras explain the Dirac postulates for the second quantized fermions.

- **Algebra in Grassmann space** offers as well the **second quantized "fermions" with integer spins** and in $d \geq 5$ the charges in adjoint representations, fulfilling the anticommutation relations without postulating them.

- **In d-dimensional Clifford space of two kinds of anticommuting coordinates γ^a's and $\tilde{\gamma}^a$'s, one has**
  $\{\gamma^a, \gamma^b\}_+ = 2\,\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,,$
  $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0 \,,$
  $(\gamma^a)^\dagger = \eta^{aa}\,\gamma^a \quad , \; (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a \,,$
  $\{S^{ab}, \tilde{S}^{cd}\}_- = 0 \,, \; \mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab} \,,$
  $a = (0, 1, 2, 3, 5, \ldots.., d)$**,**
- **The two kinds of the Clifford algebras, formed by**, γ^a and $\tilde{\gamma}^a$,
  are completely independent, **each offering** 2^d **"vectors"** ,
  which are superposition of products of either γ^a's or $\tilde{\gamma}^a$'s ,
  $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$,
- **Clifford space offers correspondingly $2 \cdot 2^d$ degrees of freedom, the same number as the Grassmann space.**

- **One can arrange each of two kinds of "vectors", formed by products of superposition of either γ^a's or $\tilde{\gamma}^a$'s into 2^d irreducible representations with respect to the corresponding Lorentz group** ,
- **making "vectors" in each of the two spaces to be"eigenvectors" of the Cartan subalgebra of the corresponding Lorentz algebras** .

$$\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\, d},$$
$$\tilde{\mathbf{S}}^{03}, \tilde{\mathbf{S}}^{12}, \tilde{\mathbf{S}}^{56}, \cdots, \tilde{\mathbf{S}}^{d-1\, d}.$$

- **Let us choose the irreducible representations of the Lorentz group to be the "eigenvectors" of each of the Cartan subalgebra members of each of the two Lorentz algebras.**

$$
\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,,\\
S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\\
\tilde{S}^{ab}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,,\\
\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,.
\end{aligned}
$$

$k = \pm\frac{i}{2}$, if either $a = 0$ or $b = 0$, $k = \pm\frac{1}{2}$, otherwise.

- k **represents an half integer spin**.

- **Let us introduce the notation for the "eigenvectors" of** **the two completely independent Cartan subalgebras**

$$
\begin{aligned}
\stackrel{ab}{(k)}: &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad \stackrel{ab}{(k)}^{\dagger} = \eta^{aa}\stackrel{ab}{(-k)}\,, \quad (\stackrel{ab}{(k)})^2 = 0\,,\\
\stackrel{ab}{[k]}: &= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad \stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,,\\
\stackrel{ab}{(\tilde{k})}: &= \frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, \quad \stackrel{ab}{(\tilde{k})}^{\dagger} = \eta^{aa}\stackrel{ab}{(-\tilde{k})}\,, \quad (\stackrel{ab}{(\tilde{k})})^2 = 0\,,\\
\stackrel{ab}{[\tilde{k}]}: &= \frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,, \quad \stackrel{ab}{[\tilde{k}]}^{\dagger} = \stackrel{ab}{[\tilde{k}]}\,, \quad (\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}\,,
\end{aligned}
$$

with $k^2 = \eta^{aa}\eta^{bb}$.

- **The "eigenvectors" of the Cartan subalgebras are either** **nilpotents** — $(\stackrel{ab}{(k)})^2 = 0$ **and** $(\stackrel{ab}{(\tilde{k})})^2 = 0$ — **or** **projectors** — $(\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}$ **and** $(\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}$, **each in its own space**.

- **Let us make a choice of the starting odd "vector" in space of odd products of γ^a's for $d = 2(2n+1)$ as follows, denoting this "vector" by $\hat{b}_1^{1\dagger}$ and its Hermitian conjugated partner by $\hat{b}_1^{1} = (\hat{b}_1^{1\dagger})^\dagger$.**
- $$\begin{aligned} \hat{\mathbf{b}}_1^{1\dagger} &= \overset{03}{(+\mathbf{i})}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2}{(+)}\;\overset{d-1\,d}{(+)}\,, \\ \hat{\mathbf{b}}_1^{1} &= (\hat{\mathbf{b}}_1^{1\dagger})^\dagger = \overset{d-1\,d}{(-)}\;\overset{d-3\,d-2}{(-)}\cdots\overset{56}{(-)}\overset{12}{(-)}\overset{01}{(-\mathbf{i})}\,. \end{aligned}$$
- **All the rest "vectors" $\hat{b}_1^{m\dagger}$, belonging to the same Lorentz representation $f = 1$, while $m = (1, 2, \cdots, 2^{\frac{d}{2}-1})$, follow by the application of the Lorentz generators S^{ab}'s. The new representations are not reachable by the Lorentz generators S^{ab} and must be found in a different way.**

Let us recognize that:

-
$$\hat{\mathbf{b}}^{\mathbf{m}}_{\mathbf{1}}\hat{\mathbf{b}}^{\mathbf{m}'\dagger}_{\mathbf{1}} = \delta^{\mathbf{mm}'} \overset{\mathbf{03}}{[-\mathbf{i}]}\overset{\mathbf{12}}{[-]}\overset{\mathbf{35}}{[-]}\cdots\overset{\mathbf{d-3\,d-2}}{[-]}\overset{\mathbf{d-1\,d}}{[+]}\,.$$

- For all the irreducible representation $f = (1, \cdots, 2^{\frac{d}{2}-1})$ of an odd products of γ^a's one finds
$(\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{d-1\,d}{[-]}\,,$
$\overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-1\,d}{[-]}\,,$
$\overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[+]}\cdots\overset{d-1\,d}{[-]}\,,$
$\cdots$
changing two by two $\overset{ab}{[-]}$ with $\overset{ab}{[+]}$.

- **Making a choice of the the vacuum state**

$$
\begin{aligned}
|\psi_{oc}> \;=\;& \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{d-1\;d}{[-]} + \\
& \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-1\;d}{[-]} + \\
& \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[+]}\cdots\overset{d-1\;d}{[-]} + \cdots |1>, \\
& \text{for } d = 2(2n+1),
\end{aligned}
$$

n is a positive integer. It follows

$$
\begin{aligned}
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f}\}_{+} |\psi_{oc}> &= \delta^{mm'} |\psi_{oc}>, \\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{+} |\psi_{oc}> &= 0\, |\psi_{oc}>, \\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{+} |\psi_{oc}> &= 0\, |\psi_{oc}>, \\
\hat{b}^{m\dagger}_{f} |\psi_{oc}> &= |\psi^{m}_{f}>, \\
\hat{b}^{m}_{f} |\psi_{oc}> &= 0\, |\psi_{oc}>,
\end{aligned}
$$

with (m, m') denoting the "family" member and (f, f') denoting "families".

- One recognizes that $\hat{b}_f^{m'\dagger}$ and $\hat{b}_f^{m}$ have all the properties of creation and annihilation operators,
  fulfilling the
  anticommutation relations of Dirac fermions,
  without postulating these relations,
  if we require that only Hermitian conjugated partners "meet" with their creation operators.
- This is an useless requirement.
- And yet we have NO quantum number of irreducible representations, which would be, however, appreciated as the family quantum number.
- The part of the Clifford space, spanned by $\tilde{\gamma}^a$'s has completely equivalent properties!!

- **To remedy these troubles let us "sacrifice" one of the two vector spaces, $\tilde{\gamma}^a$'s,**
  and use $\tilde{\gamma}^a$'s to define the "family" quantum number
  for the irreducible representation
  of the vector space of γ^a's.

  keeping the relations
  $\{\gamma^a , \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a , \tilde{\gamma}^b\}_+,$
  $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0,$
  $(\gamma^a)^\dagger = \eta^{aa}\,\gamma^a,\ (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a.$
  $(a, b) = (0, 1, 2, 3, 5, \cdots , d).$

- **Let us** *postulate***:**

$$\tilde{\gamma}^a B(\gamma^a) \;=\; (-)^B\, i\, B\gamma^a\,,$$

  with $(-)^B = -1$, if B is an odd product of γ^a's, otherwise $(-)^B = 1$.

- The vector space of $\tilde{\gamma}^a$'s has correspondingly no meaning any longer, it is "frozen out".
- The irreducible representations of Lorentz transformations, S^{ab}, have now the family quantum numbers.

  The eigenvalues besides of the operators S^{ab} also of $\tilde{S}^{ab}$ on nilpotents and projectors of γ^a's can be calculated

$$S^{ab} \stackrel{ab}{(k)} = \frac{k}{2} \stackrel{ab}{(k)}, \qquad \tilde{S}^{ab} \stackrel{ab}{(k)} = \frac{k}{2} \stackrel{ab}{(k)},$$

$$S^{ab} \stackrel{ab}{[k]} = \frac{k}{2} \stackrel{ab}{[k]}, \qquad \tilde{S}^{ab} \stackrel{ab}{[k]} = -\frac{k}{2} \stackrel{ab}{[k]},$$

  S^{ab} on nilpotents and projectors of γ^a's differ from the eigenvalues of $\tilde{S}^{ab}$.
  $\tilde{S}^{ab}$ denote the irreducible representations of S^{ab} with the "family" quantum number.

- The Lorentz invariant action for a free massless fermion in Clifford space is well known.

$$\mathcal{A} \;=\; \int d^d x \, \frac{1}{2} \, (\psi^{\dagger} \gamma^0 \, \gamma^a p_a \psi) + h.c. \, ,$$

$p_a = i \, \frac{\partial}{\partial x^a}$,

$$\gamma^a p_a |\psi> \;=\; 0 \, , p^a p_a |\psi >= 0 \, .$$

- **Solutions are for free massless "fermions" superposition of $\hat{b}^{m\dagger}_{f}$, for a chosen "family" f,**

$$|\phi^s_{fp}> \;=\; \sum_m c^{ms}{}_{fp} \, \hat{b}^{m\dagger}_{f} \, e^{-ip_a x^a} \, |\psi_{oc}> \, ,$$
$$\hat{b}^{s\dagger}_{fp} \;=\; \sum_m c^{ms}{}_{fp} \, \hat{b}^{m\dagger}_{f} \, e^{-ip_a x^a} \, ,$$

s represents different solutions of the equations of motion, $< \phi^s_{fp} | \phi^{s'}_{f'p'} >= \delta_{ss'} \, \delta_{ff'} \, \delta^{pp'}$, where I am assuming the discretization of momenta p^a.

It only remains to:

- **Recognize that the Clifford algebra offers the second quantized fermions without postulating the second quantization relations.**
- Since the states are for different momentum orthogonal, **the creation and annihilation operators fulfill the anticommutation relations for each momentum p^a.**

$$
\begin{aligned}
\{\hat{b}^{s}_{fp}, \hat{b}^{s'\dagger}_{f'p'}\}_{+}\, |\psi_{oc}> &= \delta^{ss'}\, \delta_{ff'}\, \delta_{pp'}\, |\psi_{oc}>\,,\\
\{\hat{b}^{s}_{fp}, \hat{b}^{s'}_{f'p'}\}_{+}\, |\psi_{oc}> &= 0\, |\psi_{oc}>\,,\\
\{\hat{b}^{s\dagger}_{fp}, \hat{b}^{s'\dagger}_{f'p'}\}_{+}\, |\psi_{oc}> &= 0\, |\psi_{oc}>\,,\\
\hat{b}^{s\dagger}_{fp}\, |\psi_{oc}> &= |\psi^{s}_{fp}>\,,\\
\hat{b}^{s}_{fp}\, |\psi_{oc}> &= 0\, |\psi_{oc}>\,.
\end{aligned}
$$

- **I have demonstrated that either the Grassmann – Part I – or the Clifford algebra – Part II – offer the explanation for the by Dirac assumed second quantized relations for fermions**.
- **The Grassmann algebra offers the second quantized fermions with integer spins and for** $d \geq 5$ **the charges in the adjoint representations and NO families.**
  The Clifford algebra offers families,
- **For** $d \geq (13+1)$ **the Clifford offers** also **all the charges needed to explain properties of the observed quarks and leptons**,

  as suggested by the *spin-charge-family* **theory**.
- **The** *spin-charge-family* **theory offers much more.**